Top-heavy Pyramids

March 12, 2019

Contents

1 Spoil it for me

1.1 Task 1

We start off by playing around and we find a few different THPs:

21	20	20
9 12	9 11	11 9
4 5 7	5 4 7	7 4 5

Notice that some have the same peak, even though the numbers are in different places at the base. So the position of the numbers must affect the value of the peak.

There are 6 ways to arrange the numbers in the base:

4	5	7
4	7	5
5	4	7
5	7	4
7	4	5
7	5	4

Trying out each of these, it looks like reversing the order of the numbers in the base gives the same peak, e.g.:

21	21
9 12	12 9
4 5 7	7 5 4

In fact, reversing the order of the base seems to completely flip the numbers in the triangle.

Ah, this makes sense, because the same numbers are still next to each other in the base, which means that in the second row up we'll still get the same sums. If a row is reversed, the sums are done in reverse, so the next row up is reversed as well.

That means that of the 6 ways to arrange the numbers, there's only 3 that are really different, because each one has a mirror image!

Since there's only 3 we can easily look at each of them to find the highest and lowest peaks:

```
    20            21            23
  9    11       9    12       11    12
5    4    7   4    5    7   4    7    5
```

And we can see that 20 is the lowest peak and 23 is the highest peak.

Since we have them, we might as well look and see if we can observe anything.

1.1.1 Observations

- It looks like the lowest peak occurs when the lowest base number (4) is in the middle, and the highest peak occurs when the highest base number (7) is in the middle.
- In fact, as the middle base number goes up, the peak seems to go up by the same amount: $4 \to 5 \to 7$ and $20 \to 21 \to 23$ both go up by 1 and then 2.
- The middle base number appears to be exactly 16 smaller than the peak.
- The sum of the 3 base numbers is also 16.

There's probably more that can be observed, but let's put this aside for now and move on to Task 2, and see if that helps us shed light on anything.

1.2 Task 2

First let's try the base numbers in order, why not.

```
        29
     11    18
   4    7    11
1    3    4    7
```

That's too small, but let's take a moment to assess.

Before we had 3 base numbers (for short let's call it a 3-base THP), and there were 6 ways of arranging them. But because reversing the base reversed the entire triangle, there were only 3 different peaks.

Now we have a 4-base THP, which means that there are 24 ways of arranging the numbers in the base (there are 4 choices for the first number, then there are 6 ways of arranging the remaining three numbers, in total $4 \times 6 = 24$). Reversing the base should again reverse the entire triangle — let's just sanity-check that:

29

18 11

11 7 4

7 4 3 1

Yep, it's not a proof but it's good enough for now. This means that of the 24 4-base THPs, there should be 12 that are really different. Does that mean 12 different peaks?

Let's get back to the task at hand: we're trying to make a peak of 31. In the 3-base THPs, it seemed like the number in the middle was related to the size of the peak. So assuming that that's still true, if we want a bigger peak we should put bigger numbers in the middle. Let's go for broke and try the biggest:

37

16 21

5 11 10

1 4 7 3

We overshot. But so far our hunch is holding up - bigger numbers in the middle did make the peak bigger!

But hold on, there's more than one way to put the big numbers in the middle. Let's see what happens to the peak if we do it a different way.

37

19 18

8 11 7

1 7 4 3

Huh - the peak didn't change!

Thinking back again to 3-base THPs, when we got two peaks that were the same, it turned out that one triangle was a flip of the other. That's not true in this case.

In fact, there's four ways to put the biggest numbers in the middle: the two we've got already, and their reversals (which we're pretty sure will give mirror images of the two we have). So four will have the same peak. Is this always true? It seems a bit too convenient to be a coincidence.

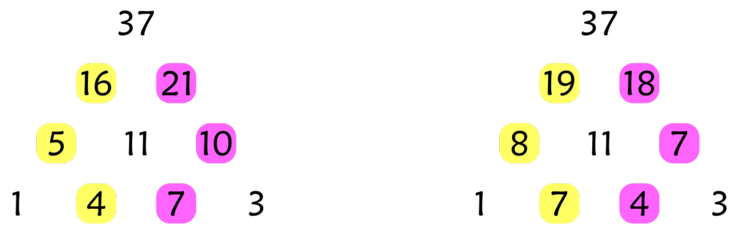
Let's pin down exactly the claim that we're making: *In a 4-base THP, reversing the inner two numbers won't change the peak.*

We haven't proved anything yet, this is just a conjecture. We have to dig a bit more.

Let's compare the two we have.

In green are the numbers that stay the same. That 11 is interesting: but oh, of course, we swapped the 4 and the 7 but they'll still sum to the same thing. So the middle number on the second row won't change when we swap the middle two in the base, no matter what the numbers are.

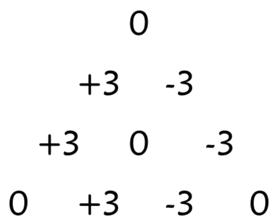
How about the numbers that changed? Let's look at them:



It looks like the numbers in yellow have each increased by 3, and the numbers in pink have each decreased by 3. If it's always true, no matter what the numbers are, that the change in the yellow ones is balanced out by the change in the pink ones, then that would explain why the peak doesn't change. But is it true?

One way to make sense of this is to consider the changes on the bottom row, and see how they affect the entries on the next rows. We can actually look at the changes by themselves, forgetting the original numbers.

On the bottom row, the changes were 0, +3, −3, 0. Let's see what they do to the rest of the triangle:



So for example on the second row up, the first entry changes by +3, because it was the sum of an entry that didn't change and an entry that changed by +3.

This is independent of the original numbers, and it's also independent of the difference between the numbers that were swapped (3 could have been any number).

So our claim is true: in a 4-base THP, swapping the middle two numbers in the base doesn't change the peak.

Ok, so there are 24 possible arrangements of the base numbers, and they come in groups of four: you can reverse the order or swap the middle two (or swap the outside two) and the peak won't change. So that means there's at most 6 different peaks, corresponding to the bases:

1	3	4	7
1	3	7	4
1	4	7	3
3	1	4	7
3	1	7	4
4	1	3	7

We also suspect that big numbers in the middle will make a bigger peak. But this isn't a very clear statement, since there are two numbers in the middle. Is 1 and 7 a "bigger" choice than 3 and 4?

We've had two different peaks so far: 29 (with 3 and 4 in the middle) and 37 (with 7 and 4 in the middle). Maybe let's just compare the two and see what else we can see.

When we write down the base we can swap the middle two and the outer two at our leisure, so let's make the two examples as similar to each other as possible:

29

12 17

5 7 10

1 4 3 7

37

16 21

5 11 10

1 4 7 3

This time we've swapped two numbers in the base that have a difference of 4, and the peak changes by 8. So does it change by double the difference?

Again let's pin down the exact claim we're making: *In a 4-base THP, if we swap an outer base number with an inner base number, the peak will change by twice their difference. The bigger peak occurs when the bigger base number is the inner one.*

We want the peak to be 31, which means we want it to be 6 smaller than the one we've got. So if our claim is true, then we could swap the 4 and the 1: putting 1 in the middle should reduce the peak, and since their difference is 3 it should reduce the peak by 6...

31

13 18

5 8 10

4 1 7 3

Bingo!

So we've got our answer to the task, but we got there by means of lucky guesses. We still don't really understand what was going on.

We can try the same approach as before: See what changed from the THP with a peak of 37 to the THP with a peak of 31. The differences were:

-6

-3 -3

0 -3 0

+3 -3 0 0

Hmm. This shows that it's true that if we swap two numbers in the base that have a difference of 3, the peak will differ by 6. No matter what the base is. And in fact it wasn't important that the difference was 3, let's try with any difference (using d to mean the difference between the two):

-2d

-d -d

0 -d 0

+d -d 0 0

So our claim was true, the peak definitely does change by double the difference. But it's still not crystal clear why. (It's a proof but not one from the book.)

Let's try a different approach. We can pick one entry of the bottom row and track it as it's added into different entries of the THP. For example the the 4 in the base of

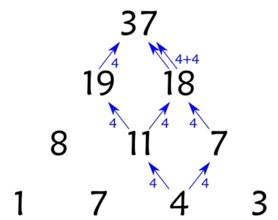
37

19 18

8 11 7

1 7 4 3

goes into the 11 and the 7 above it. Via the 11 it goes into the 19 and the 18, and via 7 it goes into 18 a second time. So by the time it reaches the peak it goes into 37 three times!



So 37 has three lots of 4 in it.

Another way to visualise this is to not do the additions as we go up, but leave them unevaluated. It uses up a lot of paper but it shows us the structure:

37	1+7+7+4+7+4+4+3
19 18	1+7+7+4 7+4+4+3
8 11 7	1+7 7+4 4+3
1 7 4 3	1 7 4 3

So 37 has 1 lot of 1, 3 lots of 7, 3 lots of 4, and 1 lot of 3. Or:

$$37 = (1 + 3) + 3 \times (7 + 4)$$
$$\text{Peak} = (\text{outside base numbers}) + 3 \times (\text{inside base numbers})$$

Now it feels like we're starting to get somewhere. Onwards!

1.3 Task 3

To start let's try the base numbers in order again.

167
70 97
28 42 55
10 18 24 31
3 7 11 13 18
1 2 5 6 7 11

That's way too small. We can try a few more arrangements:

201
111 90
59 52 38
29 30 22 16
12 17 13 9 7
1 11 6 7 2 5

218
97 121
37 60 61
11 26 34 27
3 8 18 16 11
2 1 7 11 5 6

That one on the left is pretty close!

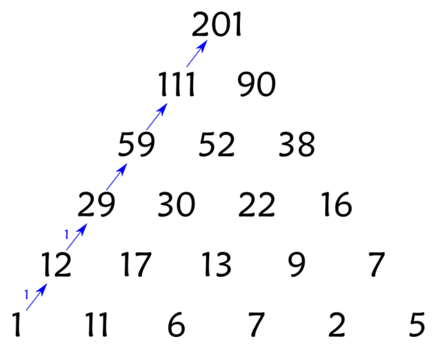
This is the sort of exercise that really makes you feel the meaning of "brute force". How many arrangements are there?

6 choices for the first position, then 5 for the second, then 4 for the third, so on... 1 for the last. So $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$. Ok, half are mirror images, so really only 360 arrangements to check. Still! We don't want to do that.

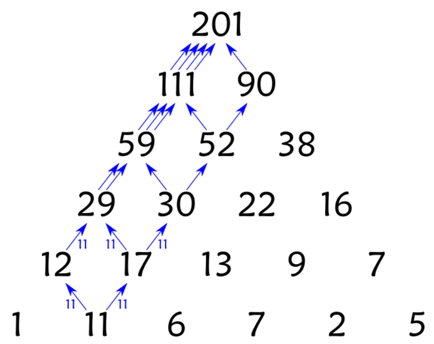
Instead, now we know a little bit about how the peak is made for the 4-base THP, let's try investigating how the peak is made for this 6-base THP.

Let's take the 201 peak THP as our example. 1 goes into 12 on the second row, which goes into

29, 59, 111, and 201. Once only, it doesn't go anywhere else.

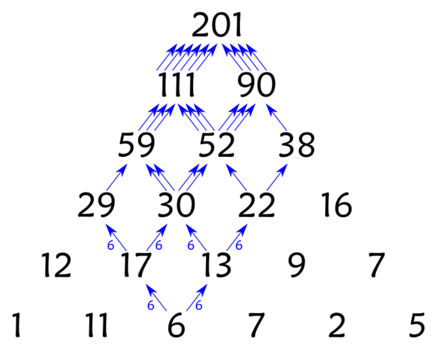


The 11 goes into the 12 and the 17 etc:



and finally goes into 201 a total of five times.

The 6 goes into the 17 and the 13 etc:



and finally goes into 201 a total of ten times.

The 7, 2 and 5 are mirror images of the first three.

So altogether, 201 is made of:

- 1 lot of 1
- 5 lots of 11
- 10 lots of 6
- 10 lots of 7
- 5 lots of 2
- 1 lot of 5

If the base numbers are different we can put them in order in the above list, sum them up and see what the peak should be.

So we can write the peak in terms of the base numbers:

$$\text{Peak} = (\text{outside numbers}) + 5 \times (\text{next numbers in}) + 10 \times (\text{middle numbers}).$$

How does this help us get a peak of 200?

It may help to stare at the above equation for a bit. The last term says that no matter what the middle numbers are, you have to multiply them by 10 to find their contribution to the peak. So this last term will always be a multiple of 10.

In a similar way, the second term will always be a multiple of 5. The peak, if it's 200, is also a multiple of 5. That means the first term should also be a multiple of 5.

So the two numbers we choose to put on the outside of the base should add to a multiple of 5.

The numbers were 1, 2, 5, 6, 7, and 11. Checking pairwise, we find that it's actually impossible to get a pair of these that add to a multiple of 5! So we can't get a peak of 200!

1.4 Conclusion

We have found that:

1. Reversing the base of the THP reverses the entire thing
2. In a 4-base THP you can swap the two middle numbers in the base without changing the peak (some of the numbers inside don't change either, but others do)
3. For a 3-base THP there's a maximum of 3 different peaks, and for a 4-base THP there's a maximum of 6 different peaks
4. Bigger numbers in the middle make a bigger peak
5. In a 3-base THP, the biggest peak occurs when the biggest number is in the middle, and the smallest occurs when the smallest number is in the middle
6. That the differences between entries in two THPs is a THP itself
7. In a 4-base THP, swapping and inner and outer base number changes the peak by their difference
8. The number of times the base numbers contribute to the peak is given by a row of Pascal's triangle
9. Used facts about multiples to show that you can't get a peak of 200 in the last example

We haven't proved too many of these. To be specific, we've explicitly proved items 2, 3, 7, and 9. Most of the rest we have pretty good arguments for, but we didn't pursue them to the point of being sure BAD.

To really see why, especially for item 8, a little bit more work is required! But we have to leave you something to do.