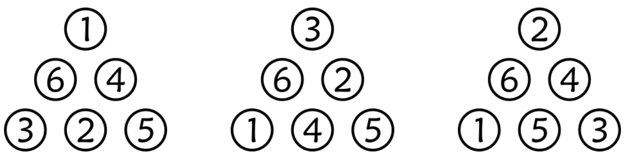
6 circles

February 14, 2019

Contents

1 Spoil it for me

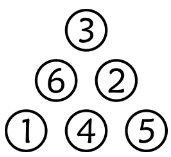
Trying out numbers in the circles, we find that there are actually quite a few solutions. Here are three, along with their side-sum S:



$S = 10$ $S = 10$ $S = 9$

Notice that there are some that have the same S, despite being "different"... are they secretly the same?

Some solutions definitely are related, e.g.



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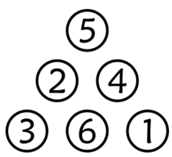
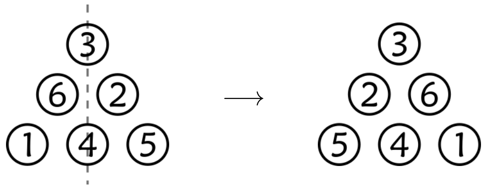
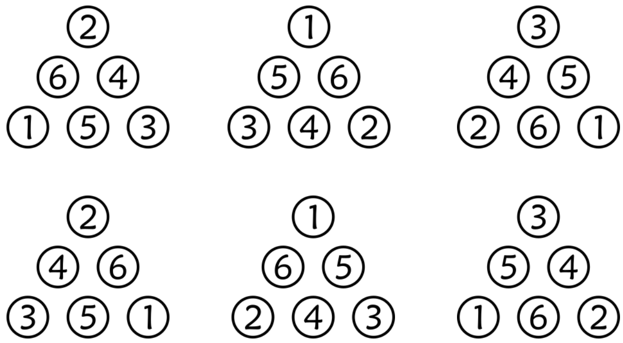


are simply rotations of each other, and $S = 10$ for both. This makes sense, because a rotation won't change the side-sum! Neither will a flip, e.g.:



So if we've found one solution, we can rotate it or flip it to get another without changing S. How many can we relate in this way?

Well, there are 3 rotations of a solution, and each one can be flipped to get another solution, so there's 6 in total that are related by rotating and flipping. For example,



These all have a side-sum of 9.

So are these all the same or are they different?

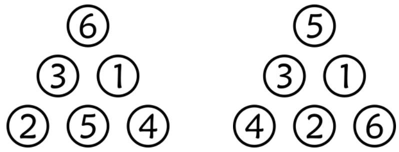
1.1 Which ones are "the same"?

Here there is a choice to be made: A choice of definition of "the same". No choice is right or wrong, but some may be more helpful than others, depending on what we're trying to do.

For now, let's say they are the same if they have the same side-sum. There may be more that give the same side-sum than the 6 you get from rotating and flipping, but we'll put that question on the shelf for now.

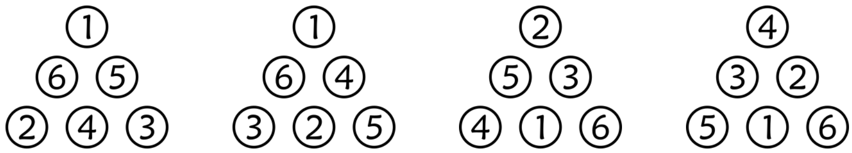
The next question is, how many different (what we're now calling different) solutions are there? In other words, how many different side-sums are there?

So far we've found solutions with side-sums of 9 and 10. Working a bit harder we can find one with $S = 11$ and one with $S = 12$:



$S = 11$ $\quad$ $S = 12$

Now we have 4 different solutions. Since we don't care about orientation let's put the smallest corner at the top and the largest at the bottom-right:



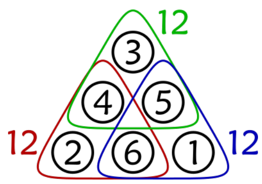
$S = 9$ $\quad$ $S = 10$ $\quad$ $S = 11$ $\quad$ $S = 12$

We don't know we've got everything yet, but we've got enough to make some observations.

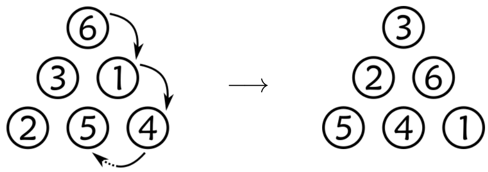
1.2 Observations

Looking at all the ones we've got, here are some things available to be noticed.

1. We haven't found a side-sum smaller than 9 or bigger than 12
2. $S = 9$ has the smallest numbers in the corners, $S = 12$ has the largest numbers in the corners
3. $S = 10$ has the odd numbers in the corners, $S = 11$ has the even numbers in the corners
4. 6 is always next to 1
5. The 3 small triangles inside each big one also sum to the same thing:

   and that total seems to be the "opposite" of whatever S is: if $S = 12$ this total is 9, if $S = 9$ it's 12, and similarly for 10 and 11
6. Some seem to be a weird sort of not-quite-rotation of each other, e.g.:

   as if they live on a bicycle chain going around a sprocket. "Sprocketing" a solution seems to turn the side-sum of 9 into 12, 10 into 11, and vice-versa
7. ...

1.3 Observation 1: Is it possible to get 8 or 13?

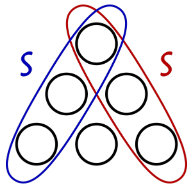
Maybe we should try to find the smallest possible side-sum. We know that 6 has to go somewhere, and if we want the smallest side-sum it makes sense to put 6 on the same side as the two smallest possible numbers, namely 1 and 2. Doing this we get a side-sum of 9, so we've shown that we can't get S smaller than 9. To show that we *can* get 9 all we need is the example we already found. So 9 actually is the smallest.

We can do a similar argument to show that 12 is the highest: 1 has to go somewhere, and to get the biggest we should put it with 5 and 6, which gives 12. And we already have an example with a side-sum of 12 that works, so 12 is the biggest.

1.4 Observations 2 & 3: What goes in the corners?

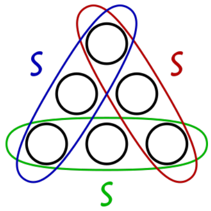
It sort of feels like the numbers in the corners contribute twice. But they don't contribute twice to the side sum. What do they contribute twice to?

Each corner contributes to two side-sums. For example, here the top corner contributes to the blue and red side-sums:



So if we add the blue and red ones together (this gives $2S$, since they're both S), the top corner will contribute twice to this.

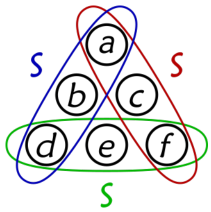
A-ha! Instead of just two, we can add all three side-sums together:



Then each corner will contribute twice to this sum of side-sums (let's call it T for total), which is

$$T = 3S.$$

Let's take stock and think about what T is made of. Now may be a nice time to bring in some notation for the values in the circles:



T is made using the middle numbers once, and the corner numbers twice:

$$T = (b + c + e) + (2a + 2d + 2f),$$

or, to put it another way, we have all the numbers once and then the corner numbers once again,

$$T = (a + b + c + d + e + f) + (a + d + f).$$

This form might be more useful to us, because while we may not know which letter corresponds to which number, we know that they are each one of 1-6. That means we know that in some order,

$$\begin{aligned} &a + b + c + d + e + f \\ &= 1 + 2 + 3 + 4 + 5 + 6 \\ &= 21 \end{aligned}$$

So that means

$$T = 21 + (a + d + f),$$

in other words, the total (sum of the side sums) is equal to 21 plus the sum of the corners!

1.4.1 Cool, but so what?

What have we actually found?

Remember that $T = 3S$, since it's the sum of the side-sums and the side-sums are equal. That means that

$$21 + (a + d + f) = 3S.$$

Let's look at this equation for a bit and see what we can see.

- This equation gives a direct relationship between the side-sum and the sum of the numbers in the corners.
- Since S is a whole number, $21 + (a + d + f)$ must be a multiple of 3. Since 21 is a multiple of 3, we come to the conclusion that the sum of the corners must be a multiple of 3.
- The smallest that a, d and f can be are 1, 2, and 3, so that $a + d + f = 6$. This gives 27 on the LHS, so $S = 9$. This tells us, in a different way from before, that 9 is the smallest side-sum.
- The largest that a, d and f can be are 4, 5, and 6, so that $a + d + f = 15$. This gives 36 on the LHS, so $S = 12$. Again we have a new way of knowing that 12 is the biggest side-sum.
- Taking the three observations above and putting them together, we have that $a+d+f$ must be a multiple of 3, but it can't be smaller than 6 or larger than 15. So the only options for $a + d + f$ are 6, 9, 12 and 15, which give side-sums of 9, 10, 11 and 12, respectively.
- Let's rearrange the equation slightly, to make S more obvious:

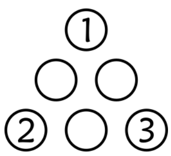
  $$S = 7 + \frac{a + d + f}{3}$$

  (This may have been helpful earlier, oh well!) With the right knowledge at hand we can realise that this says the side sum is 7 more than the average of the corners! Weird - why 7 more?

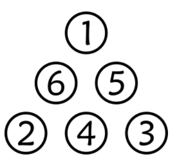
1.4.2 Constructing solutions with side-sums of 9 and 12

From the observations above, we know that if we want the smallest S, we want the smallest numbers in the corners. And this makes sense when we remember that the corners contribute twice to T (which is 3 lots of the side-sum).

Let's assume a, d, and f are 1, 2, and 3:

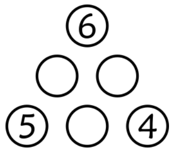


Now, *if* this can form a valid solution, it should have $S = 9$. To test it we can fill in each blank to make up 9 on each side:

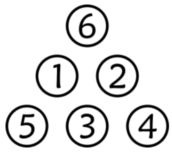


and check that we've used each number once - voila, a valid solution!

Similarly for $S = 12$ and having the biggest numbers in the corners:



To test it we fill in the blanks to make up 12:



Again, voila!

1.4.3 Constructing solutions with side-sums of 10 and 11

It felt like we relied on a bit of luck for the last two, and it worked out. (Though really we knew that it had to work, since we'd already found the solutions.)

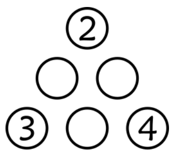
But what about $S = 10$ and 11? They're not the biggest or smallest so it's not obvious what to choose for the corners. So we'll need to be a bit more careful.

From the observations above we know that for $S = 10$, the corners should add to 9:

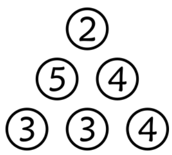
$$a + d + f = 9.$$

How can we make 9 using three of the numbers from 1-6?

We know from examples that it's possible to use $1 + 3 + 5$. But let's see if we can find another way. One way is $2 + 3 + 4$. Let's try it:

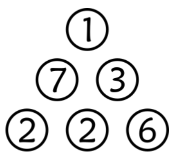


If it works we should have a side-sum of 10. So we fill in the gaps to make 10 on each side:



Darn. This isn't a valid solution, because it uses 3 and 4 twice and doesn't use 1 or 6.

Ok, so we now know that there are some choices for corners that don't work, even if they satisfy the equation we found. We could also have chosen $1 + 2 + 6$ for the corners. Since the examples we've got each seem have a nice symmetry to them (smallest, odd, even, largest) it seems unlikely that this unbalanced choice will work, but let's try it anyway. Put $\{1, 2, 6\}$ in the corners, and fill the gaps to add to 10:

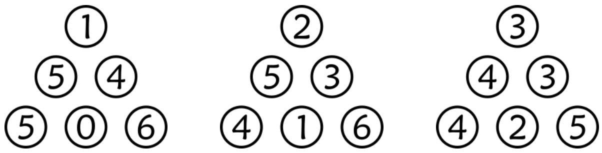


This time 2 is used twice, 4 and 5 are missing, and we have a new problem: 7 isn't allowed! So this isn't valid either. The only valid solution for $S = 10$ has 1, 3, and 5 in the corners.

Now for $S = 11$: in this case $a + d + f = 12$. There are three ways to pick three numbers from 1-6 that add to 12:

$$1 + 5 + 6, \qquad 2 + 4 + 6, \qquad 3 + 4 + 5.$$

Putting these numbers in the corners, and filling in the blanks to make sure the sides add to 11:



The first and the last are not valid solutions, and the middle is the one we've already found, so this is the only way to get $S = 11$.

1.5 Wrapping up (but not finished!)

We have managed to:

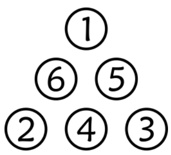
- Find four solutions to the 6-circle problem
- Recognise that they can be rotated and reflected
- Show that they are the only solutions
- Understand why there can't be any smaller or larger side-sums
- Understand why there aren't any different solutions that give the same side-sum (but maybe not in the most satisfying way)
- Understand how the side-sums relate to the choice of corners

We have not:

- Figured out if there is a deeper reason why 6 is always next to 1
- Understood why the 3 small triangles in each solution sum to the same thing
- Understood the "Sprocketing" phenomenon
- Figured out if there is a deeper reason why the side-sum is 7 plus the average of the corners

Some of these may be harder than others, some may not have solutions. But we would not be true to ourselves if we left you nothing to do!

Throughout there has been an idea that keeps popping up, that we haven't focussed on. The solutions with side-sums of 9 and 12 feel similar in some way, and 10 and 11 feel related as well. In fact, in this theme, if we take the solution for $S = 9$



and swap $1 \leftrightarrow 6$, $2 \leftrightarrow 5$, and $3 \leftrightarrow 4$, we get

⑥
① ②
⑤ ③ ④

which is the solution for $S = 12$!

So maybe there are really only two solutions to the 6-circle problem…

1.6 An alternative approach

We know we only have 6 numbers to work with, and we're trying to get 3 sets of three that sum to the same thing. So let's think about what sets of three from 1-6 there can be. Going through the options systematically, we get a large list:

Set	Total
1,2,3	6
1,2,4	7
1,2,5	8
1,2,6	9
1,3,4	8
1,3,5	9
1,3,6	10
…	

and so on. Since we're looking for ones that add to the same thing, it might be more worthwhile to arrange them by total.

Set	Total	Set	Total
1,2,3	6	1,4,6	11
		2,3,6	11
1,2,4	7	2,4,5	11
1,2,5	8	1,5,6	12
1,3,4	8	2,4,6	12
		3,4,5	12
1,2,6	9		
1,3,5	9	2,5,6	13
2,3,4	9	3,4,6	13
1,3,6	10	3,5,6	14
1,4,5	10		
2,3,5	10	4,5,6	15

Using one of these sets on a side of the triangle means putting two of the numbers in the corners and the third in the middle. Let's see how it works.

First take the side-sum of 8 as an example, our options are the sets

$$1, 2, 5$$
$$1, 3, 4$$

Beyond the fact that there are only two options to use on three sides, there is no 6, so this will not give us a valid solution.

Now let's look at a side-sum of 9. Our only options are:

$$1, 2, 6$$
$$1, 3, 5$$
$$2, 3, 4$$

For each set, we want two numbers to be in corners, so they each have to appear in a second set (different for each). This is true:

$$1, 2, 6$$
$$1, 3, 5$$
$$2, 3, 4$$

We want the other number in each set to be in the middle of a side, so that number shouldn't appear anywhere else. This is also true: the numbers that appear only once are 6, 5, and 4.

This shows us not only that there is a solution for $S = 9$, but that there is only one. We can also find solutions for $S = 10$, 11, and 12 this way.

It's also a systematic approach that may be useful if we want to take the problem any further...

1.7 Think Beyond

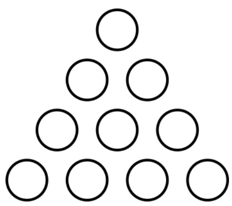
To take this further, we could:

- Use different numbers instead of 1-6 (we already have a couple of solutions if we relax this rule, that might be an interesting place to start)
- Use different numbers of circles in a triangle shape
- Use a different shape instead of a triangle

Some of these may be too relaxed, some may be too strict. Only one way to find out.

The second one is usually a fruitful track. But there's more than one way to make the triangle bigger.

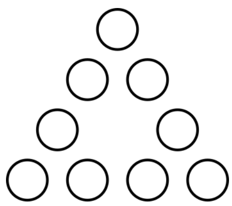
1. First choice:



using 1-10

But then we have to rethink the side-sum restriction - what about that middle circle? (Could we treat it as a garbage bin - put a number in which we then ignore?)

2. Second choice:



using 1-9

Seems a bit cleaner, and easy to see how to generalise further.

Good luck!